CONCERNING THE COMMUTATOR SUBGROUPS OF GROUPS WHOSE ORDERS ARE POWERS OF PRIMES.

BY DR. W. B. FITE.

(Read before the American Mathematical Society, September 2, 1902.)

IN the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY, volume 3 (1902), pages 331, 351, the writer has shown that the commutators of a metabelian group are invariant in the group, and that those operators of a group which correspond to the invariant operators of the group of cogredient isomorphisms are commutative with all the commutators of the group. It follows as a direct consequence of these two facts that the commutator subgroup of a group of the second or third class is abelian, that is, is of the first class. It was shown in the same article, page 349, that a metabelian group of odd order cannot be a group of cogredient isomorphisms if it has a set of generators such that the order of any one of them is not a divisor of the least common multiple of all the others. It is the main object of this paper to show that these facts are special cases of more general ones.

Let G be a group of order p^m (p being a prime) and let G' be the group of cogredient isomorphisms of G, and G'' that of G'; also let l and l' denote the classes of the commutator subgroups of G and G'' respectively. If B_1 is any commutator of G, and A_i ($i = 1, 2, \cdots, l'$) a set of any l' commutators (not necessarily distinct) of G, we have $A_i^{-1}B_iA_i = B_iB_{i+1}$. Now since the commutator subgroup of G'' is of class l' it is evident that B''_{p^r} is invariant in this subgroup (B''_{p^r} being that operator of G'' that corresponds to B_{p^r}). Therefore $B_{p^{r+1}}$ is invariant in the commutator subgroup of G and $l \leqq l'' + 1$. We have seen that for groups of classes two or three the commutator subgroups are of class one. We have therefore proved the

THEOREM: *If a group G is of order p^m (p being a prime) and class $2k$ or $2k+1$, its commutator subgroup is of class l, where $l \leqq k$.*

Suppose now that G is of class k, where $k \leqq p$, and has an abelian commutator subgroup. Let A'_i ($i = 1, 2, \cdots, n$), be a set of generators of G' of orders p^{a_i} respectively, and let A_i be

an operator of G that corresponds to A_i'. If B is any operator of G, we have $A_i^{-1}BA_i = Bt_1$, and

$$A_i^{-1}t_jA_i = t_jt_{j+1} \qquad (j=1, 2, 3, \cdots, k-1).$$

It follows from this that

$$A_i^{-p^{\alpha_i}}BA_i^{p^{\alpha_i}} = Bt_1^{p^{\alpha_i}}t_2^{\frac{p^{\alpha_i}(p^{\alpha_i}-1)}{2!}} \cdots t_j^{\frac{p^{\alpha_i}(p^{\alpha_i}-1)\cdots(p^{\alpha_i}-j+1)}{j!}} \cdots$$

$$t_{k-1}^{\frac{p^{\alpha_i}(p^{\alpha_i}-1)\cdots(p^{\alpha_i}-k+2)}{(k-1)!}} = B,$$

since A_i' is of order p^{α_i}. Also

$$A_i^{-p^{\alpha_i}}t_jA_i^{p^{\alpha_i}} = t_jt_{j+1}^{p^{\alpha_i}}t_{j+2}^{\frac{p^{\alpha_i}(p^{\alpha_i}-1)}{2}} \cdots t_{k-1}^{\frac{p^{\alpha_i}(p^{\alpha_i}-1)\cdots(p^{\alpha_i}-k+j+2)}{(k-j-1)!}} = t_j,$$

$$(j=1, 2, \cdots, k-2)$$

and $t_{j+1}^{p^{\alpha_i}} = 1$, provided $t_l^{p^{\alpha_i}} = 1$ $(l > j+1)$. Now $t_{k-1}^{p^{\alpha_i}} = 1$, and therefore $t_1^{p^{\alpha_i}} = 1$.

In a similar way we can show that the order of t_1 is a divisor of the order of B'. Also the order of $t_{j+1}(j=1, 2, \cdots, k-2)$ is a divisor of the order of t_j' and therefore a divisor of the order of B'. It follows from this that if an operator B corresponds to an operator of G' of order p^β, then it is commutative with the p^βth power of every operator of G.

Suppose now that $\alpha_1 > \alpha_i$ $(i \geqq 2)$. Then a power of A_1, less than p^{α_1}, would be commutative with every operator of G. But this is impossible.

THEOREM: *If a group G, of order p^m (p being a prime) and class $k(k \leqq p)$, has an abelian commutator subgroup, G' cannot have a set of generators such that the order of one of them is greater than the order of every one of the others.*

We now consider under what conditions a group G of order $p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$, where $p_1, p_2, \cdots, p_n$ are distinct primes, can have a commutator subgroup of order p_1^β.* Let $G_1, G_2, \cdots, G_n$ be subgroups of G of orders $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_n^{\alpha_n}$ respectively. Then $G_i(i > 1)$ is abelian, since identity is the only commutator it contains. Moreover, any operator of G that transforms G_i into

* The symmetric group of degree three is such a group. It is of order 3.2 and its commutator subgroup is of order 3.

itself is commutative with every operator of G_i. Let H_1 be the commutator subgroup. The group $\{H_1, G_i\}$ is of order $p_1^{\beta} p_i^{\alpha_i}$. This contains $p_1^{\gamma} (\gamma \leqq \beta)$ subgroups of order $p_i^{\alpha_i}$, and therefore $p_1^{\gamma} \equiv 1 \pmod{p_i}$. Hence if

$$p_1^{\gamma} \not\equiv 1 \pmod{p_i} \quad (0 < \gamma \leqq \beta),$$

every commutator is commutative with every operator of G_i. Then $A_j^{-1} A_i A_j = A_i t_{ij}$, where A_j is any operator of

$$G_j \qquad (j = 1, 2, \cdots, n)$$

and A_i is any operator of G_i; and $A_j^{-1} A_i^{p_1^{\beta_i}} A_j = A_i^{p_1^{\beta_i}}$, where $p_1^{\beta_i}$ is the order of t_{ij}. But $p_1^{\beta_i}$ is relatively prime to p_i. Therefore $A_j^{-1} A_i A_j = A_{ij}$, and G is the direct product of the groups G_j.

THEOREM. *If a group G of order $p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ ($p_1, p_2, \cdots, p_n$ being distinct primes) has a commutator subgroup of order p_1^{β} and if $p_1^{\gamma} \not\equiv 1 \pmod{p_i}$ $(0 < \gamma \leqq \beta)$, $(i = 2, 3, \cdots n)$, then G is the direct product of groups of orders $p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_n^{\alpha_n}$ respectively.*

CORNELL UNIVERSITY.

August, 1902.

NOTE ON IRREGULAR DETERMINANTS.

BY PROFESSOR L. I. HEWES.

IN Gauss's table * of binary quadratic forms the two negative determinants -468 and -931 of the first thousand are classed as regular and their genera and classes given correctly. Perott † has pointed out that these two determinants are irregular. The details of the classes of the original thirteen irregular determinants of Gauss have been worked out by Cayley ‡ and on the following page are given the details, in his notation, for the properly primitive reduced forms of the two determinants added by Perott's investigation.

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*C. F. Gauss, Werke, vol II, p 450.

† "Sur la formation des déterminants irreguliers," *Crelle*, vol. 59.

‡ Cayley's Collected Papers, vol. 5, p. 141.

D	Classes.	α	β	δ	Cp.
-468	1 0 468	+	+	+	1
	13 0 36				e^2
	9 0 52				e_1^2
	4 0 117				$e^2e_1^2$
	9 3 53	−	+	+	e_1
	17 −5 29				e^2e_1
	17 5 29				$e^2e_1^3$
	9 −3 53				e_1^3
	19 8 28	+	−	−	e
	7 −1 67				$e^3e_1^2$
	7 1 67				ee_1^2
	19 −8 28				e^3
	8 2 59	−	−	−	e^2e_1
	11 −4 44				ee_1
	11 4 44				$e^3e_1^3$
	8 −2 59				ee_1^3
					Exponent of irregularity 2.

D	Classes.	α	β	Cp.
-931	1 0 931	+	+	1
	4 1 233			g^4
	4 −1 233			g^2
	25 12 43			dg^4
	25 −12 43			d^2g^2
	11 2 85			d
	11 −2 85			d^2
	23 9 44			dg^2
	23 −9 44			d^2g^4
	19 0 49	+	−	g^3
	17 2 55			g
	17 −2 55			g^5
	5 −2 187			dg^5
	5 2 187			d^2g^5
	20 3 47			dg^3
	20 −3 47			d^2g^3
	20 7 49			d^2g
	20 −7 49			dg^5
				Exponent of irregularity 3.

KINGSTON, R. I.,
August, 1902.

NOTE ON THE PROJECTIONS OF THE ABSOLUTE ACCELERATION IN RELATIVE MOTION.

BY DR. G. O. JAMES.

INASMUCH as the usual methods of forming the projections of the absolute acceleration on the moving axes rest to a certain extent on geometric considerations it has seemed worth while to treat the question from a purely analytic point of view and derive these expressions by purely algebraic processes. This method possesses the advantage of being immediately extensible to any number of dimensions, and in the latter part of this note I shall give the expressions for n dimensions.

Connecting the moving axes Ox, Oy, Oz with the fixed axes by the following schema

	x	y	z
X	a	b	c
Y	a'	b'	c'
Z	a''	b''	c'',

I shall rapidly recall the following equations:

$$X = Sax, \quad Y = Sa'x, \quad Z = Sa''x, \tag{1}$$

where S denotes summation with respect to a, b, c and x, y, z.

$$\begin{cases} \dfrac{dX}{dt} = Sa\dfrac{dx}{dt} + Sx\dfrac{da}{dt}, \\ \dfrac{dY}{dt} = Sa'\dfrac{dx}{dt} + Sx\dfrac{da'}{dt}, \\ \dfrac{dZ}{dt} = Sa''\dfrac{dx}{dt} + Sx\dfrac{da''}{dt}; \end{cases} \tag{2}$$

$$(3)\qquad \begin{cases} \dfrac{d^2X}{dt^2} = Sa\dfrac{d^2x}{dt^2} + Sx\dfrac{d^2a}{dt^2} + 2S\dfrac{dx}{dt}\dfrac{da}{dt}, \\ \dfrac{d^2Y}{dt^2} = Sa'\dfrac{d^2x}{dt^2} + Sx\dfrac{d^2a'}{dt^2} + 2S\dfrac{dx}{dt}\dfrac{da'}{dt}, \\ \dfrac{d^2Z}{dt^2} = Sa''\dfrac{d^2x}{dt^2} + Sx\dfrac{d^2a''}{dt^2} + 2S\dfrac{dx}{dt}\dfrac{da''}{dt}; \end{cases}$$

$$(4)\qquad \begin{cases} p = \sum c\dfrac{db}{dt} = -\sum b\dfrac{dc}{dt}, \\ q = \sum a\dfrac{dc}{dt} = -\sum c\dfrac{da}{dt}, \\ r = \sum b\dfrac{da}{dt} = -\sum a\dfrac{db}{dt}. \end{cases}$$

The projections of the absolute velocity on the moving axes are then

$$(4)\qquad \begin{cases} V_x = \sum a\dfrac{dX}{dt} = \dfrac{dx}{dt} + qz - ry, \\ V_y = \sum b\dfrac{dX}{dt} = \dfrac{dy}{dt} + rx - pz, \\ V_z = \sum c\dfrac{dX}{dt} = \dfrac{dz}{dt} + py - qx. \end{cases}$$

The direction cosines satisfy the following equations

$$(5)\qquad \frac{da}{dt} = br - cq, \quad \frac{db}{dt} = cp - ar, \quad \frac{dc}{dt} = aq - bp,$$

with two similar sets in a', b', c', and a'', b'', c'', which I shall denote by (5)′ and (5)″.

The projections of the absolute acceleration on the moving axes are

$$(6)\qquad \begin{cases} J_x = \sum a\dfrac{d^2X}{dt^2} = \sum a\,Sa\dfrac{d^2x}{dt^2} + \sum a\,Sx\dfrac{d^2a}{dt^2} + 2\sum aS\dfrac{dx}{dt}\dfrac{da}{dt}, \\ J_y = \sum b\dfrac{d^2X}{dt^2} = \ldots, \\ J_z = \sum c\dfrac{d^2X}{dt^2} = \ldots. \end{cases}$$

Calculating $\sum a \delta a \frac{d^2x}{dt^2}$ and remembering the relations between the direction cosines we have at once

$$\sum a \delta a \frac{d^2x}{dt^2} = \frac{d^2x}{dt^2}. \tag{7}$$

Again

$$\sum a \delta x \frac{d^2a}{dt^2} = x \sum a \frac{d^2a}{dt^2} + y \sum a \frac{d^2b}{dt^2} + z \sum a \frac{d^2c}{dt^2}.$$

Differentiating (5) (5)′ and (5)″ and calculating

$$\sum a \frac{d^2a}{dt^2}, \quad \sum a \frac{d^2b}{dt^2}, \quad \sum a \frac{d^2c}{dt^2},$$

we have

$$\begin{cases} \sum a \dfrac{d^2a}{dt^2} = -(q^2 + r^2), \\ \sum a \dfrac{d^2b}{dt^2} = -\dfrac{dr}{dt} + pq, \\ \sum a \dfrac{d^2c}{dt^2} = \dfrac{dq}{dt} + pr. \end{cases} \tag{8}$$

Again

$$\sum a \delta \frac{dx}{dt} \frac{da}{dt} = \frac{dx}{dt} \sum a \frac{da}{dt} + \frac{dy}{dt} \sum a \frac{db}{dt} + \frac{dz}{dt} \sum a \frac{dc}{dt}$$

$$= -r \frac{dy}{dt} + q \frac{dz}{dt}. \tag{9}$$

Hence

$$J_x = \frac{d^2x}{dt^2} - (q^2 + r^2)x + \left(pq - \frac{dr}{dt}\right)y + \left(pr + \frac{dq}{dt}\right)z$$

$$- 12\left(r \frac{dy'}{dt} - q \frac{dz}{dt}\right).^* \tag{10}$$

Now (10) may be written

$$J_x = \frac{d}{dt}\left(\frac{dx}{dt} + qz - ry\right) + q\left(\frac{dz}{dt} + py - qx\right)$$

$$- r\left(\frac{dy}{dt} + rx - pz\right). \tag{11}$$

*This formula appears to have been first given by J. Liouville, *Jour. de Math. pures et appl.*, series 2, vol. 3 (1858), pp. 10–11.

Hence we have for the projections the ordinary expressions

$$(11)\qquad \begin{cases} J_x = \dfrac{dV_x}{dt} + qV_z - rV_y, \\ J_y = \dfrac{dV_y}{dt} + rV_x - pV_z, \\ J_z = \dfrac{dV_z}{dt} + pV_y - qV_x. \end{cases}$$

To extend this method to n dimensions I shall connect the moving axes $Ox_1, Ox_2, \cdots, Ox_n$ with the fixed axes OX_α, $OX_\beta, \cdots, OX_\nu$ by the following table:

	x_1	x_2	$\cdots$	x_n
X_α	a_1	a_2	$\cdots$	a_n
X_β	β_1	β_2	$\cdots$	β_n
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
X_ν	ν_1	ν_2	$\cdots$	ν_n

Then

$$(12)\qquad X_\lambda = \sum_{i=1}^{n} \lambda_i x_i,$$

$$(13)\qquad \frac{dX_\lambda}{dt} = \sum_{i=1}^{n} \lambda_i \frac{dx_i}{dt} + \sum_{i=1}^{n} x_i \frac{d\lambda_i}{dt},$$

$$(14)\qquad \frac{d^2X_\lambda}{dt^2} = \sum_{i=1}^{n} \lambda_i \frac{d^2x_i}{dt^2} + \sum_{i=1}^{n} x_i \frac{d^2\lambda_i}{dt^2} + 2\sum_{i=1}^{n} \frac{dx_i}{dt}\frac{d\lambda_i}{dt},$$

$$(15)\qquad V_{x_\tau} = Sa_\tau \frac{dX_\alpha}{dt} = Sa_\tau \sum_{i=1}^{n} a_i \frac{dx_i}{dt} + Sa_\tau \sum_{i=1}^{n} x_i \frac{da_i}{dt}$$

$$= \sum_{i=1}^{n} Sa_\tau a_i \frac{dx_i}{dt} + \sum_{i=1}^{n} Sa_\tau \frac{da_i}{dt} x_i$$

$$= \frac{dx_\tau}{dt} + \sum_{i=1}^{n} x_i Sa_\tau \frac{da_i}{dt} = \frac{dx_\tau}{dt} + \sum_{i=1}^{n} x_i p_{i\tau},$$

where

$$(16) \qquad p_{i\tau} \equiv Sa_\tau \frac{da_i}{dt} = -\, Sa_i \frac{da_\tau}{dt},$$

$$(17) \qquad J_{x_\tau} = Sa_\tau \frac{d^2 X_a}{dt^2} = Sa_\tau \sum_{i=1}^{n} a_i \frac{d^2 x_i}{dt^2} + Sa_\tau \sum_{i=1}^{n} x_i \frac{d^2 a_i}{dt^2} + 2Sa_\tau \sum_{i=1}^{n} \frac{dx_i}{dt} \frac{da_i}{dt}$$

$$= \sum_{i=1}^{n} \frac{d^2 x_i}{dt^2} Sa_\tau a_i + \sum_{i=1}^{n} x_i Sa_\tau \frac{d^2 a_i}{dt^2} + 2 \sum_{i=1}^{n} \frac{dx_i}{dt} Sa_\tau \frac{da_i}{dt}.$$

The differential equation satisfied by the direction cosine λ_i is found by expressing the fact that the point at unit distance on the axis OX_λ has a velocity zero along the moving axes. Hence

$$(18) \qquad \frac{d\lambda_i}{dt} + \sum_{j=1}^{n} \lambda_j p_{ji} = 0.$$

From (18)

$$(19) \qquad \frac{d^2 \lambda_i}{dt^2} = \sum_{j=1}^{n} \lambda_j \frac{dp_{ij}}{dt} + \sum_{j=1}^{n} \frac{d\lambda_j}{dt} p_{ij}.$$

Whence

$$(20) \qquad Sa_\tau \frac{d^2 a_i}{dt^2} = \sum_{j=1}^{n} \frac{dp_{ij}}{dt} Sa_\tau a_j + \sum_{j=1}^{n} p_{ij} Sa_\tau \frac{da_i}{dt} = \frac{dp_{i\tau}}{dt} + \sum_{j=1}^{n} p_{ij} p_{j\tau}.$$

Hence

$$(21) \qquad J_{x\tau} = \frac{d^2 x_\tau}{dt^2} + \sum_{i=1}^{n} x_i \left(\frac{dp_{i\tau}}{dt} + \sum_{j=1}^{n} p_{ij} p_{j\tau} \right) + 2 \sum_{i=1}^{n} \frac{dx_i}{dt} p_{i\tau}$$

$$= \frac{d}{dt} \left(\frac{dx_\tau}{dt} + \sum_{i=1}^{n} x_i p_{i\tau} \right) + \sum_{i=1}^{n} \left(\frac{dx_i}{dt} + \sum_{j=1}^{n} x_j p_{ji} \right) p_{i\tau}$$

$$= \frac{d}{dt} V_{x\tau} + \sum_{i=1}^{n} V_{x\tau} p_{i\tau} \qquad (\tau = 1, 2 \cdots, n).$$

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INFINITESIMAL DEFORMATION OF THE SKEW HELICOID.

BY DR. L. P. EISENHART.

(Read before the American Mathematical Society, September 3, 1902.)

CONSIDER the skew helicoid S, defined by the equations

$$x = u\cos v, \quad y = u\sin v, \quad z = av. \tag{1}$$

We shall show that the problem of the infinitesimal deformation of this surface can be completely solved.

By direct calculation we find

$$E = \sum\left(\frac{\partial x}{\partial u}\right)^2 = 1, \quad F = \sum\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} = 0,$$
$$G = \sum\left(\frac{\partial x}{\partial v}\right)^2 = u^2 + a^2, \tag{2}$$

and

$$X, Y, Z = \frac{a\sin v, -a\cos v, u}{\sqrt{u^2 + a^2}} \tag{3}$$

where Y, X, Z denote the direction cosines of the normal. Again we find

$$D = \sum X\frac{\partial^2 x}{\partial u^2} = 0, \quad D' = \sum X\frac{\partial^2 x}{\partial u\,\partial v} = \frac{-a}{\sqrt{u^2 + a^2}},$$
$$D'' = \sum X\frac{\partial^2 x}{\partial v^2} = 0. \tag{4}$$

The characteristic equation of the deformation reduces in this case to

$$\frac{\partial^2\phi}{\partial u\partial v} + \frac{u}{u^2 + a^2}\frac{\partial\phi}{\partial v} = 0,$$

of which the general integral is

$$\phi = \frac{U + V}{\sqrt{u^2 + a^2}}, \tag{5}$$

where U is a function of u alone and V is a function of v alone.

The cartesian cöordinates of the surface S_1, corresponding to S with orthogonality of linear elements, have the following expressions:*

$$(6)\qquad \begin{aligned} x_1 &= (U+V)\sin v - 2\int \sin v \cdot V'dv, \\ y_1 &= -(U+V)\cos v + 2\int \cos v \cdot V'dv, \\ z_1 &= -\frac{1}{a}[(U+V)u - 2\int u \cdot U'du], \end{aligned}$$

where the accent denotes differentiation. From (6) we have that, when V is a constant, S_1 is a surface of revolution. Moreover, since these formulæ involve an arbitrary function of u, it follows that any surface of revolution can be defined by them.

Conversely, *given a surface of revolution defined by*

$$x = u\cos v, \quad y = u\sin v, \quad z = U;$$

the helicoid with plane director, whose equations are

$$\bar{x} = U_1 \sin v, \quad \bar{y} = -U_1 \cos v, \quad \bar{z} = av,$$

has the same axis and corresponds with orthogonality of linear elements, if

$$U_1 = au\int \frac{U'}{u^2}du,$$

where the accent denotes differentiation with respect to u.

By direct calculation we find from (6),

$$(7)\qquad F_1 = \sum \frac{\partial x_1}{\partial u}\frac{\partial x_1}{\partial v} = \frac{V'}{a^2}[u(U+V) - (u^2+a^2)U'].$$

From (4) we see that the lines $u = \text{const.}$, $v = \text{const.}$ on S are asymptotic, and consequently the corresponding lines on S_1 form a conjugate system. Hence it follows from (7) that *the necessary and sufficient condition that asymptotic lines on S cor-*

* Bianchi, Lezioni, p. 276.

respond to lines of curvature on S_1 is that $V' = 0$, that is, S_1 must be a surface of revolution.

From (5) we see that in the latter case ϕ is a function of u alone. This, however, is a general property of the infinitesimal deformation of minimal surfaces. For, from the following formula, which we have established elsewhere,*

$$F_1 = F\phi^2 + \frac{1}{K^2(EG - F^2)}\left(D\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}\right)\left(D'\frac{\partial\phi}{\partial v} - D'\frac{\partial\phi}{\partial u}\right),$$

it is seen that when S is a minimal surface referred to its asymptotic lines, the necessary and sufficient condition that the parametric lines on S_1 be the lines of curvature is that ϕ shall be a function of u alone or a function of v alone.

When in particular we take

$$U = \sqrt{u^2 + a^2}, \quad V = 0, \tag{8}$$

we get

$$x_1 = \sqrt{u^2 + a^2}\cdot\sin v, \quad y_1 = -\sqrt{u^2 + a^2}\cdot\cos v,$$

$$z_1 = -a\log(u + \sqrt{u^2 + a^2}),$$

which define the catenoid. From (5) we get $\phi = 1$, which is the case whenever, in the deformation of a minimal surface, the adjoint of the latter is taken for the surface S_1.†

Genty‡ has shown that the cartesian coördinates, x_0, y_0, z_0, of the associate surface§ S_0 in an infinitesimal deformation are given by the equations

$$dx_1 = z_0dy - y_0dz,$$
$$dy_1 = x_0dz - z_0dx,$$
$$dz_1 = y_0dx - x_0dy.$$

Substituting the expressions for $x, y, \cdots, z_1$, from (1) and (6), and solving we find

$$x_0 = \frac{1}{a}[(U + V - uU')\sin v + V'\cdot\cos v],$$
$$y_0 = \frac{1}{a}[-(U + V - uU')\cos v + V'\cdot\sin v], \tag{9}$$
$$z_0 = U'$$

* *Amer. Jour. of Math.*, vol. 24, p. 177.
† *Ibid.*, p. 192.
‡ Toulouse *Annales*, vol. 9.
§ Bianchi, *l. c.*, p. 279.

The linear element of S_0 is readily found to be

$$(10)\quad ds_0^2 = \frac{U''^2}{a^2}(u^2+a^2)\,du^2 + \frac{1}{a^2}(V''+V-uU'+U)^2\,dv^2.$$

It is well known that the lines upon any associate surface corresponding to the asymptotic lines on S form a conjugate system. From (10) we see that the conjugate system on S_0 corresponding to asymptotic lines on S are the lines of curvature. Furthermore, the lines of curvature $v=$ const. are geodesics and consequently S_0 is a surface of Monge.* From the form of the coefficient of dv^2 in (10) we have that the generating developable is a cylinder.† Hence

In any infinitesimal deformation of a skew helicoid the associate surface is a moulure surface.

Conversely, given any moulure surface; its equations can be put in the form (9) and then can be taken for the associate surface in the deformation of the helicoid (1), corresponding to the value $(U+V)(u^2+a^2)^{-\frac{1}{2}}$ of the characteristic function.

From (6) and (9) we get the

THEOREM: *When the surface S_1 in an infinitesimal deformation of a skew helicoid is a surface of revolution, the associate surface S_0 also is a surface of revolution, and their lines of curvature correspond.*

And conversely,

When the associate surface S_0 is a surface of revolution, the characteristic surface S_1 is a surface of revolution.

When in particular S_1 is the catenoid, S_0 is the sphere of radius unity and center at the origin.

If we put

$$U = a\sqrt{u^2+a^2} - au\log(u+\sqrt{u^2+a^2}),\quad V=0,$$

the formulæ (9) define the catenoid. We have shown‡ that the necessary and sufficient condition that the lines of curvature be unaltered in the deformation of S is that S_0 be the adjoint minimal surface of S. Hence, when

$$\phi = a\left[1 - \frac{u}{\sqrt{u^2+a^2}}\log(u+\sqrt{u^2+a^2})\right],$$

* Monge, Applic. de l'Analyse a la Géométrie 5 ed., chap. 25.
† Darboux, Leçons, vol. 1, p. 105.
‡ L. c., p. 199.

the corresponding deformation of S leaves the lines of curvature unaltered and only in this case.

PRINCETON,
June, 1902.

ON INTEGRABILITY BY QUADRATURES.

BY DR. SAUL EPSTEEN.

(Read before the American Mathematical Society, September 3, 1902.)

THE object of this note is to show that Vessiot's noted theorem that: "the necessary and sufficient condition that a linear differential equation shall be integrable by quadratures is that its group of rationality shall be integrable," * is a special case of the Jordan-Beke † theorem on reducibility of differential equations.

The Jordan-Beke theorem is to the effect that "if a linear differential equation is reducible in the sense of Frobenius ‡ then its group of rationality will transform a certain linear manifoldness of the solutions (which does not include the total n-dimensional manifoldness) into itself."

Analytically interpreted § this says that the group

$$
\begin{aligned}
y_1 &= a_{11}y_1 + \cdots + a_{1k}y_k,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
y_k &= a_{k1}y_1 + \cdots + a_{kk}y_k,\\
y_{k+1} &= a_{k+1,1}y_1 + \cdots + a_{k+1,k}y_k + a_{k+1,k+1}y_{k+1} + \cdots + a_{k+1,n}y_n,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
y_n &= a_{n1}y_1 + \cdots + a_{nk}y_k + a_{n,k+1}y_{k+1} + \cdots + a_{nn}y_n,
\end{aligned}
\tag{1}
$$

is isomorphic with the group of rationality. For convenience it is well to adopt Loewy's notation, writing for (1) simply the coefficients

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* Vessiot: *Ann. de l'Ec. nor. sup.*, 1892.

† C. Jordan. *Bull. de la Soc. Math. de France*, vol. 2; Beke: *Math. Annalen*, vol. 45, p. 279.

‡ Frobenius: *Crelle*, vol. 76.

§ A. Loewy: "Ueber die irreduciblen Factoren," etc., *Berichte der math.-phy. Classe der Königl. Sächs. Gesellschaft der Wissenschaften zu Leipzig*, vol. 54 (1902), pp. 1–13.

$$(2)\qquad \left|\begin{array}{ccc|ccc} a_{11} & \cdots & a_{1k} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{k1} & \cdots & a_{kk} & 0 & \cdots & 0 \\ \hline a_{k+11} & \cdots & a_{k+1,k} & a_{k+1,k+1} & \cdots & a_{k+1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1} & \cdots & a_{nk} & a_{n,k+1} & \cdots & a_{nn} \end{array}\right|.$$

When a linear differential equation

$$(3)\qquad \frac{d^n y}{dx^n} + p_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + p_n y = 0$$

is given, we fix a domain of rationality and now if equation (3) reduces within this domain to

$$(4)\qquad \frac{d^k y}{dx^k} + q_1 \frac{d^{k-1} y}{dx^{k-1}} + \cdots + q_k y = 0 \qquad (k < n)$$

the group (2) will be isomorphic with the group of rationality of the equation (3).

If $k = n - 1$, then by effecting a quadrature equation (3) will reduce to

$$(5)\qquad \frac{d^{n-1} y}{dx^{n-1}} + r_1 \frac{d^{n-2} y}{dx^{n-2}} + \cdots + r_{n-1} y = 0$$

and the group (2) will become

$$(6)\qquad \begin{array}{cccc} a_{11} & \cdots & a_{1,n-1} & 0 \\ \cdot & \cdot & \cdot & \cdot \\ a_{n-1,1} & \cdots & a_{n-1,n-1} & 0 \\ a_{n,1} & \cdots & a_{n,n-1} & a_{nn}. \end{array}$$

If now (5) reduces within the same domain of rationality to

$$(7)\qquad \frac{d^{n-2} y}{dx^{n-2}} + s_1 \frac{d^{n-3} y}{dx^{n-3}} + \cdots + s_{n-2} y = 0$$

the group (6) becomes

$$
(8)\qquad
\begin{array}{llll}
a_{11} & \cdots a_{1,n-2} & 0 & 0 \\
\cdot & \cdot \quad \cdot \quad \cdot & \cdot & \cdot \\
a_{n-2,1} & \cdots a_{n-2,n-2} & 0 & 0 \\
a_{n-1,1} & \cdots a_{n-1,n-2} & a_{n-1,n-1} & 0 \\
a_{n,1} & \cdots a_{n,n-2} & a_{n,n-1} & a_{nn}.
\end{array}
$$

When equation (3) is integrable by quadratures this process can be continued n times; the group we are considering (which is isomorphic with the group of rationality) takes the form

$$
(9)\qquad
\begin{array}{llll}
a_{11} & & & \\
a_{21} & a_{22} & & \\
a_{31} & a_{32} & a_{33} & \\
\cdot & \cdot \quad \cdot & \cdot & \cdot \\
a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1} \\
a_{n,1} & a_{n,2} & \cdots & a_{n,n-1} \quad a_{n,n}.
\end{array}
$$

But (Lie-Engel, Transformationsgruppen I, chapter 27) the group (9) is an integrable group.

The Jordan-Beke condition for reducibility, employed above, is both necessary and sufficient. We have thus deduced as a special case of the Jordan-Beke theorem the theorem of Vessiot, namely, "*the necessary and sufficient condition that a linear homogeneous differential equation shall be integrable by quadratures is that its group of rationality be integrable.*"

PHILADELPHIA, PA.
August, 1902.

THE CENTENARY OF THE BIRTH OF ABEL.

AT the close of the first week in September last, the Royal Frederick University at Christiania, Norway, celebrated the one hundredth anniversary of the birth of Niels Henrik Abel. The occasion was noteworthy as the first international academic celebration in Norway and was in every way instructive and enjoyable for the delegates and guests of the university. On their arrival at the station they were met by representatives of the university who had been previously instructed as to which language each of them spoke and who conducted them to the rooms to which they had been assigned in the various hotels,

acting the while as interpreters or informants and talking with apparent indifference French, German, English, or Norwegian. Each delegate was presented with two large volumes of which one was: Norway, the official publication of the country at the time of the Exposition at Paris in 1900, and the other: Niels Henrik Abel, Mémorial publié à l'occasion du centenaire de sa naissance, containing an ode to Abel by Bjoernson, a historical account of his life and correspondence by Holst, the letters written or received by Abel translated into French with notes and with facsimile reproductions, letters and documents relative to him likewise with notes, the original text of such letters as Abel wrote in Norwegian, and an essay by Sylow on Abel's studies and discoveries with further facsimiles. It is this volume which for most of us will probably be the most cherished and permanent monument of the celebration. There are three other volumes now publishing which may be mentioned here as memorials of the occasion and which will doubtless be of a greater value from a purely scientific point of view than those presented to us by the university. These are volumes XXVI, XXVI *bis*, XXVI *ter*, of the *Acta Mathematica*, which will contain recently discovered and hitherto unpublished memoirs by Abel and original memoirs by a great number of mathematicians upon those branches of mathematics in which Abel was especially interested.

Although there was an informal reception for the delegates on Thursday evening, September 4, in St. Hanshaugen Park with an introductory address of welcome by Professor Nansen, chairman of the Abel committee, the official opening of the celebration took place Friday at noon under the presidence of the rector of the university and the president of the academy of sciences. The programme consisted of a beautiful cantata composed especially for the occasion by Bjoernson and Sinding, of addresses by the minister of state and the rector of the university with responses by Professors Heinrich Weber and Volterra on behalf of the delegates, and of an address or essay by Professor Sylow on the work of Abel. Professor Sylow alone spoke in Norwegian and copies of his address translated into French were distributed among the delegates.

In the evening his majesty, King Oscar II, who in person attended the official meetings of the celebration, tendered a supper to the delegates and other invited guests. The long informal receptions before and after the supper afforded the first

and probably the best opportunity which we had to become acquainted with one another.

On Saturday at noon the second official meeting was held under the auspices of the university. This was more of an academic, less of a state affair, than that on the preceding day. The president of the academy of sciences delivered an address. Forsyth, Gravé, Picard, Schwarz, Zeuthen responded on behalf of the delegates, Hensel for *Crelle's Journal*, Mittag-Leffler for the *Acta Mathematica*. The delegates then filed by and presented the inscribed congratulatory addresses or the words of felicitation sent by the various societies or universities represented.

It is difficult to obtain an exact list of the delegates, for some, Peano, Pincherle and Padé, for example, who were expected and who would have added to the already brilliant assemblage, did not arrive; and others, as Selivanov and Knott, that one could notice came unofficially or too late to be included in the lists. The correspondents of the press give the following list taken down as the delegates filed by.

From Germany and Austria: Schwarz, H. Weber, Pringsheim, Hensel, Hilbert, Seeck, Kiepert, Fr. Meyer, Krause, Engel, Gutzmer, Lerch, Czuber. From Belgium: Mansion, Paige. From Holland: Cardinaal, Kristensen. From Denmark: Thiele, Zeuthen. From France: Picard and E. Sauvage. From America: Canfield, Olsen, Spangler, Watson (not mathematicians representing various universities), and Fields, Newcomb, Van Vleck, Wilson (the last two representing the AMERICAN MATHEMATICAL SOCIETY). From Switzerland: Graf, Fehr. From Greece: Stephanos. From Russia and Finland: Lindelöf (senior), Donner, Tallqvist, Bäcklund, Zdanov, Gravé, Zantshevsky. From Sweden and Norway: A. B. Bäcklund, Wijkander, Möbius, Théel, Bendixson, von Koch, Dunér, Mittag-Leffler, Dahle, Hansen, Lugaard, Spaare. From Italy: Orland, Volterra, Del Pezzo. From Great Britain: Lamb, Love, Forsyth, Joly, Sampson, Greenhill, Hobson, Jack.

The rector of the university then delivered an address preliminary to conferring the honorary degrees. By the original charter of the university the power of bestowing such degrees had been withheld. For this celebration, however, special permission had been granted and to signalize the occasion further a new degree, Doctor Mathematicæ, was introduced. This distinctive degree was bestowed upon the following twenty-nine

persons of whom the ten who were present went forward as their names were read and received their diplomas amid great applause:

Appell, Bäcklund, Boltzmann, G. Cantor, Cremona, Darboux, G. Darwin, Dedekind, Dini, Forsyth, J. W. Gibbs, Hilbert, Jordan, Lord Kelvin, Klein, Koenigsberger, Markov, Mittag-Leffler, Newcomb, Picard, Poincaré, Lord Rayleigh, Salmon, Schwarz, Stokes, Volterra, Weber, Wirtinger, Zeuthen.

Professor Newcomb made a short speech thanking the university on behalf of the recipients of the degree. The rector of the university delivered a peroration and declared the celebration officially at an end.

But for the delegates much was yet in store — perhaps the pleasantest part of all. Professor Bjerknes and his son took some of us who were especially interested in mechanics, among whom could be seen Volterra, Love, and Lamb, to his laboratory and showed us his apparatus specially devised for exhibiting his hydrodynamic action at a distance. In the evening the city of Christiania gave us a dinner, during and after which we had another excellent opportunity to renew or further our acquaintance with one another. This occasion was the climax of the celebration. A magnificent collation was served. The King and the various nationalities were toasted. Enthusiasm waxed to fever heat. Finally the students came in a torchlight procession under the windows of the banquet hall, greeting the delegates with prolonged and uproarious applause. On Sunday various excursion parties into the environs of Christiania were formed for our interest. Sunday evening there was a special representation of Ibsen's Peer Gynt at the national theater to which we had invitations. For the following Wednesday Professor Mittag-Leffler extended an invitation to lunch to such of us as might be in Stockholm.

By Monday evening most of us had left Christiania feeling that the celebration had been uncommonly successful: for the committee had accomplished the difficult task of making each succeeding event better than the previous one.

E. B. WILSON.

PARIS, *September* 25.

THE ENGLISH AND FRENCH TRANSLATIONS OF HILBERT'S GRUNDLAGEN DER GEOMETRIE.

1. *Les Principes fondamentaux de la Géométrie.* Par D. HILBERT. Traduit par L. LAUGEL. Paris, Gauthier-Villars, 1900. 4to. 111 pp.

2. *The Foundations of Geometry.* By DAVID HILBERT. Translated by E. J. TOWNSEND. Chicago, The Open Court Publishing Company, 1902. 8vo. 132 pp.

IT is indeed a matter for congratulation that Professor Hilbert's masterly discussion of the foundations of geometry has become so well known and so widely circulated. This circumstance is undoubtedly due to its clearness and force; for, while some of the previous studies along similar lines are difficult even for the advanced student to understand, Hilbert's style is so deceivingly clear as to lead certain minds to predict the use of this book in elementary instruction.

An excellent review of the original,* rendered into English by Professor Ziwet, was given in the BULLETIN (volume 6, 1900, pages 287–299) by Dr. Sommer, of Göttingen; the present review will therefore not deal with the German edition, except for purposes of comparison.

If a minute criticism of the language of the French translation were the main purpose, the reviewer would certainly feel great hesitation in undertaking the task. But there are certain additions to the French translation which are most noteworthy, and with these we shall occupy ourselves chiefly.

On page 291 of his review, mentioned above, Dr. Sommer explains that Hilbert's axiom V is not sufficient to furnish a satisfactory foundation for the complete discussion of "the continuity of the straight line in the ordinary sense." This lack is supplied in the French edition by an additional axiom (by Hilbert) entitled "Axiome d'intégrité" (Vollständigkeitsaxiom), which practically requires that the system already set up shall

*In this connection, a review of Hilbert's Grundlagen der Geometrie by H. Poincaré, *Bull. des Sciences Math.* (2), vol. 26 (Sept., 1902), p. 249, should be mentioned. This review is, as would be expected, of the greatest importance. The present writer regrets that his review was in type before that of Poincaré was seen. Footnotes have been added occasionally at points where this review would have been most influenced by Poincaré's.

be considered as closed, *i. e.*, incapable of extension by the adjunction of additional elements. This axiom corresponds to the one introduced by Hilbert (*Jahresbericht der Deutschen Mathematiker Vereinigung*, 1900) for arithmetic; and is certainly radically different, in the character of the assumption made, from the other axioms proposed by Hilbert. Its introduction enables us to prove the fundamental theorem of Bolzano, but the reviewer feels that its form is scarcely final, and that it is not wholly satisfactory to Hilbert himself.*

Another addition to the French edition merits more attention than it seems to have received; the original conclusion of the German edition is replaced by a new appendix (pages 106–111), written by Hilbert, which shows clearly the importance of his work for non-euclidean geometry in its broadest sense, a point upon which Hilbert dwelt to a considerable extent in his lectures preceding the publication of the original Festschrift, but which is scarcely mentioned in the Festschrift itself. The apparent reason for the tendency of non-euclidean geometers in the past to cast out the axiom of parallels alone, is that this was thought to be the only axiom *explicitly* assumed by Euclid which could not be directly tested by physical experiment.† Another axiom, or set of axioms, namely those regarding continuity, assumed *implicitly* by Euclid, is, however, equally open to philosophical exception.‡ The effect of the omission of the axiom of Archimedes (Hilbert, V) was therefore of peculiar interest to Hilbert, and the subject was discussed considerably in his lectures. A student, Dr. Dehn, completed these inquiries, and a review (by Hilbert) of Dr. Dehn's thesis forms the major portion of this addition to the French translation. The one striking fact that *a geometry is possible in which the sum of the angles of any triangle is two right angles and in which similar non-congruent triangles exist, while an infinity of parallels to any straight line may be drawn through any point*, will sufficiently indicate the remarkable nature of Dr. Dehn's results.

This geometry is called "semi-euclidean," and is evidently non-archimedean, the axioms III and V being omitted; but in the whole discussion the axioms I, IV and II (in modified form) are assumed to hold. To the reviewer it would seem, from the theorem just mentioned, that the assumption that the

* See also Poincaré, l. c., pp. 256, 271, 272.

† The reviewer disclaims any intention of saying that this is a logically valid reason; it is merely the *actual* reason.

‡ See Poincaré, l. c., pp. 250, 258.

sum of the angles of a triangle is two right angles would be more advisable than the form of the parallel axiom given by Euclid (Hilbert III), since it is demonstrated that it is a "weaker" axiom, in the sense of Professor Moore. For while it can be proved from the axiom III, without the use of axiom V, we must include V in order to prove III from it. At any rate, as Dr. Dehn remarks, this substitute for the axiom of parallels is conclusively shown not to be equivalent to Euclid's form; and, while Euclid probably deserves no praise in the matter, his was certainly a happy choice in that he possessed Archimedes's axiom only by implicit assumption. While this subject merits considerable discussion, we cannot enter into further details without passing from the book in hand. Still, Dr. Dehn's thesis (*Mathematische Annalen*, volume 53 (1900)), as also his subsequent papers in the *Annalen*, form a direct and important addition to Hilbert's memoir.

Aside from these additions, the French edition is simply a remarkably good translation. The translator has happily given himself just enough liberty to render the spirit of the German into idiomatic French; but the reviewer feels confident that the sense has always been perfectly preserved. It is especially noteworthy that M. Laugel had the direct and enthusiastic assistance of Hilbert himself in the preparation of the French edition.

In spite of the hesitation expressed above, the reviewer feels moved to enquire whether better translations for "Axiome der Verknüpfung" (axiomes d'association), and for "Axiome der Anordnung" (axiomes de distribution), could not have been found, especially in view of the necessity of using these same terms in describing the corresponding sets of axioms of arithmetic in Chapter III, where the associative, distributive, and commutative laws of arithmetic are all placed under the head of "associative" axioms, whereas the "distributive" axioms are of an entirely different nature.* Finally the translations of "flächengleich" (égaux par addition), and of "inhaltsgleich" (égaux par soustraction), are certainly widely divergent from the original, and might possibly have been improved upon; but it may be contended that they are justifiable as a closer description of the ideas represented.

Of the typography nothing need be said, further than that the book is printed by the firm of Gauthier-Villars.

*Poincaré, l. c., uses "axiomes projectifs" and "axiomes de l'ordre." He expressly discards "axiomes de connection" as a translation of "Axiome der Verknüpfung."

On the whole, on account of the additions mentioned above, and in view of the faithful reproduction of the spirit of the original, the French edition compares very favorably with the German, for those who understand both languages equally well. The reviewer had excellent success in the use of the French edition by a portion of his class last winter, the students having been advised to procure either the French or the German editions at their discretion.

The English translation is the work of one who had the advantage of hearing Hilbert's preparatory lectures. It contains the additions to the French translation, but there are no new additions. Some unimportant omissions are made, including the original conclusion (also omitted in the French edition). In a short preface Professor Townsend expresses his own views upon the important features of the work. It is unfortunate that he does not emphasize the radical departures of Hilbert's *method* of presentation — especially the power obtained by considering the axioms as fundamental, the conceptions of point, line, and plane as trivial.* Notice is taken in footnotes of some recent memoirs commenting on Hilbert's work.

Some features of the translation are particularly happy. Especially does the rendering of "Widerspruchslosigkeit" by "compatibility" appeal to the reviewer. The translations of "flächengleich" (of equal area), and of "inhaltsgleich" (of equal content), §18, etc., are certainly nearer the original than the expressions used by the French translator. Where custom has established a translation of a standard technical word, Professor Townsend generally appears to have used it, as, for example, "set" for "Menge," "enumerable" for "abzählbar," "(number) field" for "(Zahl) körper," to all of which excellent alternate expressions are, however, in common use. The expression "numerical multiplicity," translated on page 126 from the French "multiplicité numérique," suffers on the other hand by comparison with Ziwet's† "manifold of numbers" as an equivalent for the German word "Zahlenmannigfaltigkeit."

*To illustrate: Hilbert, in his lectures, used effectively a geometry in which "points" are the ordinary positive integers, and "lines" are the ordinary negative rational numbers. The principle involved was, of course, used before Hilbert by others, but it has taken a far more tangible form in Hilbert's hands. See Poincaré, l. c., pp. 250, 252, etc.

†In his translation of Sommer's review above mentioned.

Professor Halsted * informs us that the translation of "Anordnung" *must* be "arrangement," but the reviewer feels that several other words are also suitable translations of "Anordnung," in its technical sense, just as "ausgezeichnet," when used technically, need not necessarily be rendered "remarkable." † The translations "order," "arrangement," and "order of sequence" are to be found in Professor Townsend's edition, and any one of them, used consistently, would have been acceptable. Professor Ziwet, in the review mentioned above, uses "order." Perhaps even a still better word might be found, but Professor Townsend is to be congratulated in not having followed the French translator in his unhappy choice. The omission of the heading "Erklärung" can be defended, although the reviewer personally inclines to its retention.

The work is unfortunately somewhat marred by several errors, some of which, occurring at critical points, are apt to be misleading. On page 24, line 11, the word "all" is wholly wrong, as the translator might have known from Hilbert's lectures or at least from Dr. Sommer's review. It may be admitted that the original is not entirely clear on this point, but no such word as "all" occurs in the German edition. The translation of "einzig" by "definite" in line 8, page 53, may possibly be confusing; *any* single point is to determine a segment 0. The word "only" must be inserted between "which" and "such" on line 11 from the bottom of page 98. On page 107, line 1, we are not to understand that Pascal's theorem is *identical* with the commutative law, but merely that the truth of the commutative law insures that Pascal's theorem holds, under the given conditions.‡ On page 124, line 11, we understand that the extraction of the square root of f_3 is the third in order, *not* that it is now to be performed "a third time." The statement on page 125, that "every regular polygon can be

* In a brief review of Professor Townsend's edition, *Science*, August 22, 1902. The review is not especially appreciative and dwells on several mistakes and misprints. Some of the rather harsh criticisms are quite unfounded, as, for instance, those regarding axiom IV, 1, p. 12; theorem 16, p. 22; and a sentence on page 25, lines 26–28; while others seem trivial, for example, that of "emanating," p. 13, etc. The errors on pp. 24 and 125, noticed by Professor Halsted, and the omission of the heading "Erklärung," are considered later in the present review. The remainder of Professor Halsted's criticisms seem unimportant, in the sense that the passages in question are at least never misleading as they stand, and certainly do not call for separate reconsideration here.

† An accepted translation for "ausgezeichnete Untergruppe" is "self-conjugate subgroup."

‡ See Poincaré, l. c., p. 267.

constructed by the drawing of straight lines and the laying off of segments" will scarcely mislead any one. It is conceivable that this curious mistranslation was caused by some oversight, but it seems that "jene" has been mistaken for "jede(s)": "Every regular polygon" should, of course, be replaced by "the above-mentioned regular polygons."

But the most dangerous errors occur in the translation of the important new appendix of the French edition. On page 127, lines 17, 19, 20, 21, "par" is translated "from" instead of "by," and on page 130, line 11, "vis-a-vis de" is also rendered "from" instead of "with respect to," while on page 128, line 9, "imposée" is changed into "confined." A reading of the paragraphs in question will disclose the fact that at least the first two of these mistakes totally destroy all meaning at critical points. These errors, and some minor ones, make this conclusion the poorest part of the English edition.

The translator's English sometimes lacks the true idiomatic ring, the worst instances being, perhaps, the footnote on page 85, and the sentence which begins on line 7, page 124. There are also several minor faults of translation, which will fortunately not mislead the student. Among these are changes from singular to plural nouns and *vice versa*, for instance, "square root(s)," line 23, page 121; the insertion and omission of words, as "consequently" line 21, page 127; and mistranslations of words. Of such minor errors the reviewer has noticed, not counting duplicates, nineteen.

Among the misprints found only two are serious: the reference to "§ 30" on page 98, line 24, should be "§ 27"; and "and" three lines from the bottom of page 123, should be "any." The misprint (or error) in the original in formula (1), § 33, corrected in the Errata (page 111) of the French edition, is corrected properly on page 104 of Professor Townsend's translation.

The word "all" on page 74, line 22, while not strictly a mistranslation of language, gives rise to an impression which is faulty, and which is not intended in the original. For it is nowhere proved that the assumption of each of the axioms IV, 1–5, is necessary for the proof of Desargues's theorem. The proper impression would possibly be conveyed by substituting the phrase "or the entire set of axioms of congruence" for the corresponding phrase in the text; the idea being that, whereas we have assumed axioms IV, 1–5, it is *necessary* to *add* to

these axiom IV, 6, but it is not shown that we need *retain* each of the axioms IV, 1–5.

The notation for a segment joining two points, *e. g.*, 0 and $\frac{1}{2}$, has been changed in § 17, perhaps for the better, from $0\frac{1}{2}$ to $(0, \frac{1}{2})$. There are also other slight changes of notation, which are generally commendable. But a great deal is lost by not following Hilbert's italicization of words and of some of the theorems, which enabled him to emphasize what he considered most important. In particular the words "NON PAS" of the principal theorem of the new conclusion to the French edition, which Hilbert and Laugel emphasized not only by extraordinary type, but also by extraordinary position in the sentence, are rendered on the last line of page 128, by an ordinary "not," in ordinary type.

The Open Court Publishing Company deserves praise for continuing to publish translations of foreign scientific classics into English. The book is well printed, except for a few bad figures, of which the worst are figures 18 (page 41) and 38 (page 76), the latter being actually misleading. The type is good, excepting the identity sign. With regret we are compelled to notice, however, that the edition is not wholly satisfactory on account of the errors mentioned above. In its present form it can scarcely be recommended to students unless they can read neither German nor French; and then it should be used only under competent guidance. But most of the serious errors could be corrected by changing a few words, and if a page of such corrections were inserted in the volume it could be used by students with greater safety.

To other than university graduate students *no* edition of the book appeals. For the statement* that it has pedagogical value, or that it is to be of influence on elementary instruction, only means that students of very advanced mathematics, by their knowledge of it, may reflect into elementary books or teaching which they may influence, something of its general spirit, almost nothing of its actual contents. To insert the system of axioms proposed by Hilbert in an elementary (high school) book or, for instance, to try to introduce into such a text-book the theory of motion as based upon congruence, or even the proof that all right angles are equal, would be as ill considered as to expect the average high school teacher to grasp the meaning of the original in its entirety.

*Sommer, l. c., p. 299; Townsend, preface to English edition.

We must rejoice, however, in proof of the wide circulation of Hilbert's ideas, that both a French and an English translation have actually been published. A widely diffused knowledge of the principles involved will do much for the logical treatment of all science and for clear thinking and clear writing in general.

E. R. HEDRICK.

YALE UNIVERSITY,
September, 1902.

DICKSON'S LINEAR GROUPS.

Linear Groups with an Exposition of the Galois Field Theory. By L. E. DICKSON, Assistant Professor of Mathematics in the University of Chicago. Teubner's Sammlung von Lehrbüchern auf dem Gebiete der mathematischen Wissenschaften mit Einschluss ihrer Anwendungen, Volume VI. Leipzig, B. G. Teubner, 1901. 8vo, x + 312 pp.

SHORTLY after the appearance of the first few numbers of the Encyclopädie der Mathematischen Wissenschaften the publishers announced a series of text-books on advanced mathematics to be issued in connection with the Encyclopädie. While the authors of articles in the Encyclopädie were especially requested to take advantage of this series to develop their subjects more fully and thus make them more accessible to the student, other writers were asked to assist to make the series as complete as possible. More than fifty different volumes of this series have already been announced, by almost as many different writers of various countries.

Never before has there been such extensive collaboration to bring the developments in the various parts of our subject within the reach of the student. It is hoped that this series will do much towards increasing the number of well-equipped investigators and thus exert a strong influence towards more substantial progress in various directions. The fact that the authors belong to so many different countries emphasizes the cosmopolitan element in mathematical work and the absence of national prejudices among its devotees.

The present work is the sixth volume of the series and is devoted to a subject which has been developed principally on French and American soil. The fundamental ideas are due to Galois and were published by him at the early age of eighteen

in a memoir entitled "Sur la théorie des nombres." * In this brief memoir Galois introduced a new kind of imaginaries and studied their classification and reduction to the least possible number.

Let $F(x)$ represent an integral function with integral coefficients. If it is possible to find three other integral functions with integral coefficients $F_1(x)$, $F_2(x)$, $F_3(x)$ such that the degrees of $F_1(x)$ and $F_2(x)$ are less than the degree of $F(x)$ and that

$$F(x) = F_1(x)F_2(x) + pF_3(x)$$

then $F(x)$ is said to be reducible modulo p. Otherwise it is said to be irreducible. In the latter case Galois represents a root of the congruence $F(x) \equiv 0 \bmod p$ by i and considers the general expression

$$a + a_1 i + a_2 i^2 + \cdots + a_{n-1} i^{n-1},$$

where $a, a_1, a_2, \cdots, a_{n-1}$ represent integers and n is the degree of $F(x)$. He observes that the $p^n - 1$ values of this general expression, when the coefficients assume separately all the least positive residues modulo p such that not all are simultaneously equal to zero, are powers of a single one of them. Hence all the algebraic quantities which enter into the theory of these imaginaries are roots of the equations of the form $x^{p^n} = x$ and are therefore independent of the form of the special irreducible congruence of degree n.†

From these results it is evident that the p^n values of the general expression of the preceding paragraph constitute a finite domain of rationality, endlicher Körper, or finite field. The interest in this theory has been greatly enhanced by Moore's proof of the fact that the elements of every finite field may be put in a (1, 1) correspondence with those of such a Galois field. ‡ The great simplicity of this field appears perhaps more remarkable when compared with the difficulties which are met in the study of other finite systems, such as Kronecker's modular systems and the theory of groups of finite order.

The first part (pages 3–71) of the present work is devoted to an exposition of the Galois field theory, chiefly in its abstract

* *Bulletin des Sciences de M. Férussac*, vol. 13 (1830), p. 428. Reprinted in *Liouville* vol. 11 (1846), p. 398.

† Oeuvres de Galois, *Liouville*, vol. 11 (1846), p. 400.

‡ This term seems to have been first used by Moore, BULLETIN, vol 3 (1893), p. 73.

form. Beginning with the simplest example, classes of residues with respect to a prime modulus, the author points out the characteristic properties of a Galois field. The greater part of the first chapter is devoted to the proof that no other finite field can exist and that there is only one such field of a given order p^n. The notation and general method of proof are practically the same as those employed by Moore in his paper entitled "A doubly infinite system of simple groups."*

The second chapter is devoted to a proof of the existence of a Galois field of order p^m, $GF[p^m]$, for every prime p and every integer m. This existence follows directly from the fact that there is at least one congruence of degree m which is irreducible modulo p. Instead of proving the latter fact directly as Serret has done,† the author assumes the existence of a $GF[p^n]$ and proves that there are irreducible functions of *every* degree in this field. Since the field of integers modulo p is known to exist, this not only proves the point in question but also exhibits some important properties of the $GF[p^{mn}]$ with respect to the included $GF[p^n]$. A somewhat different method of proof is indicated in the "exercises" which close the chapter.

The determination and classification of irreducible quantics forms the subject matter of the third chapter. Just as in the study of least residues with respect to a prime modulus, it is convenient to make prominent use of the law of exponents in the study of the irreducible quantics. A quantic $F[x]$ of degree m and irreducible in the field of order p^n is denoted by $IQ[m, p^n]$. It is said to belong to the exponent e provided e is the least positive integer for which $F(x)$ divides $x^e - 1$ in the $GF[p^n]$. When $e = p^{nm} - 1$ then $F(x)$ is said to be a primitive irreducible quantic of degree m in $GF[p^n]$. The determination of such quantics forms one of the most difficult problems in the Galois field theory. The latter part of the chapter is devoted to special methods of solving this problem.

Chapter IV is devoted to miscellaneous properties of Galois fields and begins with the study of squares, not-squares, and mth powers in a Galois field. To some readers this part would doubtless have been clearer if they were told explicitly that they were concerned with operators of a cyclic group C. The statement that "every mark has one and only one square root in the $GF[2^n]$" would thus appear as a special case of the theorem

* Moore, Published Papers of the Am. Math. Soc., vol. 1 (1896), p. 208.
† Serret, Algèbre supérieure, vol. 2 (1885), p. 137.

that in an abelian group of odd order every operator is the square of one and only one operator. It is true the zero element of $GF[2^n]$ is not contained in C but the theorem evidently applies to it. Moreover, it may be observed that 0 and 1 are the only two marks of $GF[2^n]$ which coincide with their square roots.

Similarly, Section 62 is equivalent to saying that all the operators of a subgroup of even index are squares of operators of C. The theorem of Section 63 is clearly true for any abelian group and is a direct consequence of the facts that in an abelian group there is a subgroup whose order is any arbitrary divisor $[(p^n-1)/d]$ of the order of the group and if each operator of an abelian group is raised to a power prime to the order of the group the resulting operators may be put in a (1, 1) correspondence with the original operators. In other words, every operator of an abelian group is the mth power of just one of its operators, where m is any number prime to the order of the group.

In such developments it is always a question what an author should presuppose. If a knowledge of the fundamental abstract properties of abelian groups had been assumed on the part of the reader it would have been possible at various places to indicate more clearly the contact with abstract group theory and also to bring out additional facts in the same amount of space notwithstanding the fact that the zero element in the Galois field requires separate treatment from this standpoint.

Sections 64–67 are devoted to generalizations of Jordan's results in regard to the solutions of certain quadratic equations in a Galois field. This is followed by a study of the additive groups and their multiplier Galois fields as employed by Moore. A set of m marks $\lambda_1, \lambda_2, \cdots, \lambda_m$ belonging to the $GF[p^n]$ and linearly independent with respect to the $GF[p]$ give rise to p^m distinct marks

$$c_1\lambda_1 + c_2\lambda_2 + \cdots + c_m\lambda_m \qquad (c_i = 0, 1, \cdots, p-1)$$

with respect to the larger field. These are said to form an additive group of rank m with respect to the $GF[p]$. In particular, the $GF[p^n]$ may be regarded as an additive group of rank n. These conceptions are generalized and a condition for the linear independence of marks with respect to an included field is developed. The chapter closes with a consideration of

Newton's identities for sums of powers of the roots of an equation belonging to a Galois field.

In this chapter the term *group* is used for the first time but no definition is given. The statement "since the *sum* of any two of these p^n marks may be expressed as one of the set, they are said to form an additive group" cannot be regarded as a definition. Moreover, it is apt to mislead the reader since such incomplete definitions of group are not uncommon in the mathematical literature. On the other hand, if the reader is supposed to be familiar with the definition of the term it would have been of interest to contrast this additive group with the multiplicative group formed by the marks of a field, which differ from zero.

The fifth and last chapter of Part I is devoted to the analytic representation of substitutions on marks of a Galois field. This subject is treated quite fully in Part I of the author's dissertation * where extensive references are given. Beginning with the definition of a substitution quantic the author develops the analytic conditions which characterize such functions and gives a table of all such reduced quantics whose degree is less than 6. A study of the Betti-Mathieu group and six suggestive exercises furnish the close of these interesting and important developments of modern algebra.

The second part of the present volume (pages 75–310) is devoted to a study of the most important properties of linear groups in a Galois field. Some of these groups were investigated in the field of integers modulo p by Galois, Serret and Jordan. These results are here generalized for the larger field and new systems are investigated ab initio in this field. The work of Moore first emphasized the importance of employing the general Galois field in linear group problems as the investigations are generally not much more complicated and the results are more general. Jordan frequently indicated these generalizations without entering upon their complete development.

A great part of the text is taken directly from the author's numerous memoirs, but in other parts the method of presentation differs widely from that employed in the original papers. More stress seems to have been laid upon clearness in presentation and some of the longer calculations have been avoided. The work has thus gained considerable in attractiveness and

* Dickson, *Annals of Mathematics*, vol. 11 (1897), p. 65.

has become more suitable to serve its purpose as an introduction to the extensive subject of linear groups in a finite field.

The first chapter begins with two definitions of the general linear homogeneous group on m indices with coefficients in the $GF[p^n]$. The group is denoted by $GLH(m, p^n)$ while the symbol $GLH[m, p^n]$ is used to represent its order. The latter is proved to be

$$(p^{nm}-1)(p^{nm}-p^n)(p^{nm}-p^{2n})\cdots(p^{nm}-p^{n(m-1)}).$$

For $n=1$ the factors of composition were determined by Jordan in his Traité des Substitutions. For the general value of n they were determined independently by the author and by Burnside. The general case is considered here, and from these factors it follows directly that the group of all linear fractional substitutions $LF(m, p^n)$ in the $GF[p^n]$ on $m-1$ variables and having the determinant either unity or some mth power in the field, is simple except in the two special cases $(m, p^n)=(2, 2)$ or $(2, 3)$.

The abelian linear group and a generalization of this group form the subjects of Chapters II and III. The former has been investigated by Jordan in the field of integers, while the latter is mentioned by him but he did not investigate its properties. Both have been studied by Dickson in the general Galois field and are here presented in this general way. The term abelian was first applied to these groups by Jordan, although they are not, in general, commutative. To distinguish them from commutative groups they are called abelian *linear* groups.

The hyperabelian group consists of the totality of linear homogeneous substitutions in the $GF[p^{2n}]$ which leave absolutely invariant the function

$$\psi \equiv \sum_{i=1}^{m} \begin{vmatrix} \xi_{2i-1} & \xi_{2i} \\ \xi_{2i-1}^{p^n} & \xi_{2i}^{p^n} \end{vmatrix}.$$

It was first studied by the author in the *Proceedings of the London Mathematical Society*, volume 31, and forms the subject of Chapter IV of the present volume. It is distinct from the hypoabelian groups studied by Jordan in his Traité and by Dickson in the BULLETIN and elsewhere. Moreover, it must be distinguished from Picard's hyperabelian group of infinite order. The totality of its substitutions whose coefficients belong to the included $GF[p^n]$ constitute the special abelian

linear group $SA(2m, p^n)$. The latter is transformed into itself by a subgroup of the hyperabelian group whose order is $p^n + 1$ times the order of $SA(2m, p^n)$.

The greater part of Chapter V is taken directly from the author's article entitled "The structure of the linear homogeneous groups defined by the invariant $\lambda_1\xi_1^r + \lambda_2\xi_2^r + \cdots + \lambda_m\xi_m^r$." * Chapter VI is devoted to a new exposition of the theory of compounds of a linear homogeneous group. This theory is entirely due to the author, having been introduced by him in several papers published in 1898.† He also employed it in his new definition of the general abelian linear group, which was published in the first volume of the *Transactions* of the AMERICAN MATHEMATICAL SOCIETY.

In Chapters VII and VIII the linear homogeneous group in the $GF[p^n]$ defined by a quadratic invariant is investigated. The case when $p > 2$ is considered in the former and that when $p = 2$ in the latter of these chapters. By employing the theory of compounds, developed in the preceding chapter, the investigations, especially when $p > 2$, have been presented in a much simpler manner than in the original papers. Chapter VIII is followed by twelve exercises with a number of suggestions, covering the text of the first eight chapters of this part.

The next three Chapters (IX–XI) are devoted respectively to linear groups with certain invariants whose degree exceeds 2, canonical form and classification of linear substitutions, and operators and cyclic subgroups of the simple linear fractional group $LF(3, p^n)$. In Chapter XII the subgroups of the linear fractional group $LF(2, p^n)$ are studied. The first statement of the last paragraph of this chapter is evidently incorrect. For instance, the icosahedral group may be represented as a transitive substitution group of degree 12, but it does not contain a complete system of 12 conjugate subgroups. If a simple group is represented on the smallest possible number of letters it must be primitive, and a primitive group of degree N must always contain a complete system of N conjugate subgroups; but this is not necessarily true of imprimitive groups of degree N even if they are simple.

The developments of Chapters XIII and XIV are quite different from those which precede. In the former a number

* Dickson, *Math. Annalen*, vol. 52 (1899), p. 561.

† Dickson, BULLETIN, vol. 5 (1898), p. 120; *Proc. Lond. Math. Soc.*, vol. 30 (1898), p. 70.

of theorems in regard to abstract groups are developed. Beginning with the well known theorems of Moore with respect to the abstract definitions of the alternating and the symmetric groups and the classic theorem due to Hamilton and Dyck in regard to the abstract definition of the icosahedral group, the author develops a number of other theorems with respect to the generational relations of known groups. Some of these are employed in the study of the group of the equation for the 27 straight lines on a general surface of the third order, which is the subject of Chapter XIV. The treatment of this subject seems to be especially commendable.

The study of linear groups has thus far been the most fruitful in increasing our knowledge of simple groups. As in Jordan's Traité des Substitutions, so in the present work a great deal of attention is given to the factors of composition of the groups under consideration. In this way a number of systems of simple groups are established and the closing chapter is fittingly devoted to a summary of the known simple groups. With a few exceptions all of these groups are included in one or more known infinite systems. It may be observed that the simple group of order 7920 cannot be represented on less than 11 letters so that 9 should be replaced by 11 in the list on page 309.

The present work seems to be the first separate volume on the subject to which it is devoted. The treatment of the first part has much in common with Chapters III and IV of Serret's Algèbre supérieure, volume 2 (1885), while the second part seems to have been mostly influenced by Jordan's Traité des Substitutions (1870). The author's thorough acquaintance with his subject, and his experience as a writer, have enabled him to present a somewhat abstract subject in an unusually attractive manner. It may be hoped that the book will do much to create a wider interest in these fields of higher algebra. While the generality of treatment calls for considerable maturity on the part of the reader, yet not much special knowledge is presupposed.

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STANFORD UNIVERSITY.

BUCKINGHAM'S THERMODYNAMICS.

Theory of Thermodynamics. By EDGAR BUCKINGHAM, Ph.D. New York, The Macmillan Company, 1900. xi + 205 pp. Price, $1.90.

OLDFASHIONED, or, rather, elementary, thermodynamics deals mainly with the relations of heat and mechanical energy. It considers purely physical changes, including changes from any one to any other of the three states, solid, liquid and gaseous, but does not undertake to discuss chemical changes or such chemico-physical changes as occur in solution. Accordingly, from the purely mathematical point of view, elementary thermodynamics is shamefully simple. What possibilities of interest for the mathematician can be found in a beggarly array of only five variables, p, v, T, ϵ and η, subject to one characteristic equation and two general laws, so that only two of them at a time are capable of arbitrary variation? But from the standpoint of physical interpretation and application the study of thermodynamics, even in its elements, presents respectable difficulties, and is likely to be regarded by the beginner, however mathematical he may be, as something of a mystery.

On the other hand, when we undertake to deal with chemical changes, and find ourselves confronted by "i phases" of matter, each phase containing "a mixture of K substances," the variables, "normal" and "inverse," "internal" and "external," independent and dependent, begin to manœuver in regiments at the will of the commander, and the born mathematician finds himself in his proper element.

The book before us is intended for the help of those not very rare individuals who have some difficulty with both the physics of elementary thermodynamics and the mathematics of its broader generalizations. It is not for the beginner, although it deals with the very elements and discusses the two general laws at length. It is too circumspect, too precise, too thoroughgoing, in its examination for the student who is taking his first look at the field with which it has to do. It is rather for him who, after some months of growing acquaintance with the elementary facts and laws of thermodynamics, finds himself in doubt as to whether he fully understands, for example, the

nature of a reversible cycle or the theory of the "porous plug" experiments by means of which the "absolute thermodynamic scale of temperature" was brought to concrete existence. The very great care which the author uses in examining the premises of his science is shown by the following quotations. "It is usual to speak of this temperature [the temperature indicated by a thermometer in a mixture of liquids or gases] of the mixture as being also the temperature of the separate components, and this mode of expression does not lead us into any practical difficulties, although it has evidently no justification a priori" (page 2); "Our first idea of heat is that of something which increases the temperature of a body when added to it and decreases the temperature when taken away [a footnote here excludes melting, evaporation, etc., from immediate consideration]. The first addition which we make to this conception is the *assumption*, that *when a given body cools through a definite interval, the quantity of heat it gives out is always the same*, regardless of what becomes of the heat after it leaves the body in question" (page 9). Such care is admirable and in the main reassuring, even when it results in stamping as an assumption what most readers would be likely to take as an axiom. In at least one case, however, the author's caution seems to the reviewer too great; namely, on pages 9 and 10, where he raises the question, "Are two quantities of heat, which are equal to a third, equal to each other?" and disposes of it by the "definition," "that *quantities of heat which are equal to the same quantity are equal to one another.*" It is true that the author has been feeling his way to the conception that heat is "a quantity in the ordinary sense," but after he has got so far as to say that two quantities of heat are each equal to a third, it is too late for the need of any argument or definition to the effect that these two quantities are equal to each other.

The following definition, "Any process, of which the direction may be reversed by infinitely small modifications of the outside actions, is called a REVERSIBLE PROCESS," and, in general, everything that the author has to say about reversible or non-reversible processes is luminous, precise, and much to be commended.

In a word, even the reader who does not care for the chemical applications of thermodynamics, and who has no stomach for generalized coördinates, will find much in this book that he

can profit by, though it is not written for him alone or for him primarily.

But what of him whose ambition is to go further? For him there is an excellent brief chapter on The Conditions of Thermodynamic Equilibrium, and another on Thermodynamic Potentials and Free Energy. These are preceded by the necessary disquisition on the relations and functions of the indefinitely large number of variables which may be involved, and they are followed by a chapter of applications in which are discussed the electromotive force of a reversible galvanic cell, the equilibrium of phases of a single substance (triple point, etc.), and the phase rule of Gibbs. The discussion of the last subject disclaims the intention of being "either complete or rigorous"; but as an introduction "'tis enough, 'twill serve." There is, probably, no other book so well suited as this to the needs of him who is making preparation for an attempt to explore those tremendous abysses of thought, where reigns that condition of supernal calm known as the Equilibrium of Heterogeneous Substances.

But, alas! like the fellow who had never learned to read, and who found that no spectacles, however well contrived, would enable him to read, the reviewer gazes in vain through the medium of Dr. Buckingham's book at many a passage like the following in the famous third volume of the *Transactions of the Connecticut Academy*, "*the stability of any phase in regard to continuous changes depends upon the same conditions in regard to the second and higher differential coefficients of the density of energy regarded as a function of the density of entropy and the densities of the several components, which would make the density of energy a minimum, if the necessary conditions in regard to the first differential coefficients were fulfilled.*"

Through Nature to Gibbs should be the watchword of every devout thermodynamician, and if Dr. Buckingham has not succeeded in making the way easy, the failure is not his fault.

EDWIN H. HALL.

CAMBRIDGE, MASS.,
July 23, 1902.

NOTES.

CERTAIN numbers of the *Bulletin of the New York Mathematical Society* are now nearly out of print. The committee of publication is especially desirous of procuring copies of volume 2, nos. 6 and 7, and volume 3, nos. 3 and 9, and is willing to pay one dollar per copy for these numbers. The committee respectfully requests the members of the Society to look over their files and to notify the committee of any duplicates of the numbers mentioned with which they are willing to part. The Society receives a considerable revenue from the sales of sets of the BULLETIN and is often able to secure desirable exchanges of back volumes of other mathematical journals in return for its own. These two interests will be seriously affected as soon as the present small supply of the numbers in question is exhausted, unless the committee of publication is able to replace them.

THE twelfth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held in the Ryerson Physical Laboratory of the University of Chicago, on Friday and Saturday, January 2 and 3, 1903. Titles and abstracts of papers should be in the hands of the secretary of the section, Professor T. F. Holgate, 617 Hamline St., Evanston, Ill., at least two weeks before the meeting.

THE second regular meeting of the San Francisco Section of the AMERICAN MATHEMATICAL SOCIETY will be held in room 21, North Hall, of the University of California, on Saturday, December 20, 1902. Titles and abstracts of papers should be in the hands of the secretary of the section, Professor G. A. Miller, Stanford University, at least two weeks before the meeting.

A NEW edition of the Annual Register of the AMERICAN MATHEMATICAL SOCIETY is now in preparation and will be issued in January next. Forms for furnishing necessary information have been sent to each member. Those who have not already responded to the circular are requested to do so at once.

THE October number (volume 4, number 1, second series) of the *Annals of Mathematics* contains: "The geodesic lines

on the anchor ring," by G. A. BLISS; "Proof of a theorem concerning isosceles triangles," by H. F. BLICHFELDT; "An elementary exposition of Frobenius's theory of group characters and group determinants," by L. E. DICKSON; "Communication concerning Mr. Ransom's mechanical construction of conics," by E. V. HUNTINGTON.

CAMBRIDGE UNIVERSITY.—Advanced mathematical courses for the current academic year are announced as follows:—

Michaelmas term, 1902.—By Professor Sir G. G. STOKES: Hydrodynamics, including viscosity, three hours.—By Professor A. R. FORSYTH: Differential geometry; general theory of curves and surfaces, three hours; Fourier's and other expansion theorems, two hours.—By Professor G. H. DARWIN: Orbits of planets; Figures of equilibrium of rotating fluid, three hours.—By Professor Sir R. S. BALL: Planetary theory, three hours.—By Dr. E. W. HOBSON: Spherical and cylindrical harmonics, three hours.—By Mr. J. LARMOR: Electrodynamics, three hours.—By Mr. H. F. BAKER: Elementary theory of functions, three hours.—By Mr. H. M. MACDONALD: Waves (especially waves of light), three hours.—By Mr. H. W. RICHMOND: Analytic geometry of curves, three hours.—By Mr. G. T. WALKER: The electromagnetic field, three hours.—By Mr. G. B. MATHEWS: Algebraic functions (elementary).—By Mr. A. N. WHITEHEAD: Application of symbolic logic to Cantor's theory of aggregates, three hours.—By Mr. J. H. GRACE: Invariants and geometric applications, three hours.

Lent term, 1903.—By Professor Sir G. G. STOKES: Physical optics, three hours.—By Professor A. R. FORSYTH: Differential geometry; general theory of curves and surfaces (continued), three hours.—By Professor G. H. DARWIN: Outlines of dynamical astronomy, three hours.—By Dr. E. W. HOBSON: Sound and vibrations, three hours.—By Mr. J. LARMOR: Electrodynamics, with optical and thermodynamic applications, three hours.—By Mr. H. W. RICHMOND: Analytic geometry of three dimensions, projective properties, three hours.—By Mr. H. F. BAKER: Theory of functions.—By Mr. H. M. MACDONALD: Hydrodynamics, three hours.—By Mr. G. B. MATHEWS: Theory of algebraic numbers, three hours.—By Mr. R. A. HERMAN: Hydrodynamics.—By Mr. A. BERRY: Elliptic functions.—By Mr. G. T. WALKER: The electromagnetic theory of light, three hours.—By Mr. G. T. BENNETT: Linear and quadratic complexes, three hours.—By Mr. E. T. WHIT-

TAKER: The theory of definite integrals, two hours. By Mr. J. H. GRACE: Invariants and geometric applications (continued), three hours.—By Mr. HUDSON: Geometric theory of ordinary differential equations.

Easter term, 1903.—By Professor Sir R. S. BALL: Perturbation of cometary orbits, three hours.—By Mr. W. L. MOLLISON: Theory of potential and electrostatics, three hours.—By Mr. A. N. WHITEHEAD: Non-euclidean geometry.

Long vacation, 1903.—By Professor Sir R. S. BALL: Applications of modern geometry to dynamics, three hours (short course).—By Mr. E. T. WHITTAKER: General dynamics, with applications to astronomy and thermodynamics, three hours.

THE Belgian Academy at Brussels, announces as its prize subject for 1903, the following:

"An important contribution to the study of mixed forms in any number of variables, with the application of the results to the geometry of any spaces."

THE London Mathematical Society has awarded its De Morgan Medal for 1902 to Professor A. G. GREENHILL, of the Ordnance College, Woolwich.

ON December 15, the University of Klausenberg will celebrate the hundredth anniversary of the birth of JOHANN BOLYAI.

PROFESSOR K. HENSEL, of the University of Berlin, has been called to a professorship of mathematics at the University of Marburg.

MR. ALFRED T. DE LURY, M.A., formerly lecturer in mathematics in the University of Toronto, has been appointed associate professor in the same department. DR. J. C. FIELDS has been appointed special lecturer in mathematics in the University of Toronto.

PROFESSOR L. E. DICKSON, of the University of Chicago, is now associated with Professor B. F. FINKEL, of Drury College, as editor of the *American Mathematical Monthly*. Mr. J. M. COLAW, formerly associate editor of the *Monthly*, has retired from the editorial staff.

DR. J. W. MILLER has been appointed instructor in mathematics and astronomy at Lehigh University.

DR. C. N. HASKINS has been appointed instructor in mathematics at the Massachusetts Institute of Technology.

PROFESSOR XAVIER ANTOMARI, editor of the *Nouvelles annales de mathématiques*, died at Paris, on June 9, 1902.

PROFESSOR ERNST SCHRÖDER, of the department of mathematics in the Polytechnic school in Carlsruhe, died on June 17, 1902.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BETTI (E.). Opere matematiche, pubblicate per cura della R. Accademia dei Lincei. (In 2 volumes.) Vol. I. Milano, Hoepli, 1902. 4to. Fr. 25.00

COMPTE RENDU du deuxième congrès international des mathématiciens, tenu à Paris, du 6 au 12 août 1900. Procès-verbaux et communications, publiés par E. Duporcq, secrétaire général du congrès. Paris, Gauthier-Villars, 1902. 8vo. 456 pp. Fr. 16.00

DUPORCQ (E.). See COMPTE RENDU.

DZIWINSKI (P.). Wyklady matematyki, Kurs I. Zasady geometrji analitycznéj i analizy wyzszéj, tom I. Wstep do geometrji i teorja wyznaczników. Poczatki analizy wyzszéj i zasady rachunku róznicz kowego. Lwów, 1902. 4°. 19 + 928 pp. M. 30.00

GODEFROY (M.). Théorie élémentaire des séries (limites; séries à termes constants; séries à termes variables; fonction exponentielle; fonctions circulaires; fonction gamma). Avec une préface de L. Sauvage. Paris, Gauthier-Villars, 1903. 8vo. 8 + 266 pp. Fr. 8.00

HEDRICK (E. R.). On the sufficient conditions in the calculus of variations. 8vo. (*Bulletin of the American Mathematical Society* (2) 9, pp. 11–24.)

IGURBIDE (J. F.). Nature harmonique de l'espace; traduit de l'espagnol. Barcelone, Giro, 1902. 8vo. 245 pp.

MONTCHEUIL (M. DE). Sur une classe de surfaces. (Thèse.) Paris, Gauthier-Villars, 1902. 4to. 83 pp.

RÉPERTOIRE bibliographique des sciences mathématiques. 2e série. Fiches 1001 à 1100. Paris, Gauthier-Villiars, 1901. 18mo. Fr. 2.00

SAPOLSKY (L.). Ueber die Theorie der relativ-Abelschen cubischen Zahlkörper. (Diss.) Leipzig, Teubner, 1902. 8vo. 7 + 481 + 6 pp., 35 plates. M. 6.00

SAUVAGE (L.). See GODEFROY (M.).

SIMON Y MAYORGA (J.). Caracteres de irracionalidad de los números enteros. Trabajo de investigación. Salamanca, Calon, 1902. 8vo. 20 pp. Fr. 2.00

STUYVAERT. Etude de quelques surfaces algébriques engendrées par des courbes du second et du troisième ordre. (Thèse.) Gand, Hoste, 1902. 8vo. 74 pp.

WEINNOLDT (E.). Leitfaden der analytischen Geometrie. Leipzig, Teubner, 1902. 8vo. 6 + 80 pp. Cloth. M. 1.60

WHITTAKER (E. T.). A course of modern analysis. An introduction to the general theory of infinite series and of analytic functions, with an account of the transcendental functions. Cambridge, Macmillan & Bowes, 1902. 8vo. Cloth. 12s. 6d.

II. ELEMENTARY MATHEMATICS.

ARRIAGA (M.). Geometría elemental y superior, con ejercicios de cálculo y de dibujo lineal. Barcelona, Araluce [1902]. 8vo. 241 pp.

BERGMEISTER (H.). Geometrische Formenlehre für Mädchen-Lyzeen. Teil 1: für die I. und II. Klasse. Wien, Deuticke, 1902. 8vo. 4 + 88 pp. Cloth. M. 1.50

BETTAZZI (R.). Aritmetica razionale ad uso dei ginnasi. Torino, Tipografia Salesiana, 1902. 8vo. 180 pp. Fr. 2.00

BRIOT (C.). Tratado de algebra elemental y superior. Traducción de C. Sebastián y B. Portuondo, corregida, revisada y ampliada por B. Portuondo. Madrid, Hernando, 1901. 8vo. 732 pp. Fr. 14.00

BRUNN (J.). Vierstellige Logarithmen, für den Schulgebrauch zusammengestellt. Münster, Aschendorff, 1902. 8vo. 18 pp. M. 0.25

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